CONTRIBUTIONS TO THE EARTH'S OBLIQUITY RATE, PRHCESS1ON, AND NLJTATION

James G. Williams

Jet Propulsion Laboratory, California Institute of '1'ethnology, Pasadena, CA, 91109

I. OGOS::JGW
JGW@LOGOS.JPL.N ASA.GOV

5 Tables
1 Figure

Abstract

The precession and nutations of the Earth's equator arise from solar, lunar, and planetary torques on the oblate Earth. The mean lunar orbit plane is nearly coincident with the ecliptic plane. A small tilt out of the ecliptic is caused by planetary perturbations and the Earth's gravitational harmonic J2. These planetary perturbations on the lunar orbit result in torques on the oblate Earth which contribute to precession, obliquity rate, and nutations while the $J2$ perturbations contribute to precession and nutations. Small additional contributions to the secular rates arise from tidal effects and planetary torques on the Earth's bulge. The total correction to the obliquity rate is -0,024 "/century, it is an observable motion in space (the much larger conventional obliquity rate is wholly from the motion of the ecliptic, not the equator), and it is not accounted for in the IAU-adopted expressions for the orientation of the Earth's equator. The $J2$ effects have generally been allowed for in past precession and nutation theories. For the planetary effect, the contributions to the 18.6-yr nutations are –0.03 mas (milliarcseconds) for the in-phase $\Delta\psi$ plus out-of-phase contributions of 0.14 mas in $\Delta\psi$ and –0.03 mas in $\Delta\varepsilon$. The latter terms demonstrate that out-of-phase contributions can arise by means other than dissipation. The sum of the contributions to the precession rate are considered and the inferred value of the moment of inertia combination (C–A)/C, which is used to scale the coefficients in the nutation series, is evaluated. Using an updated value for the precession rate, the rigid-body (C–A)/C = 0.0032737634 which, in combination with a satellite-derived J2, gives a normalize polar moment of inertia $C/MR^2 = 0.3307007$. The planetary contributions to the precession and obliquity rates are not constant for long times causing accelerations in both quantities. Acceleration in precession also arises from tides. Contributions from the improved theory, masses, ecliptic motion, and measured values of the precession rate and obliquity are combined to give expressions (polynomials in time) for precession , obliquity, and Greenwich Mean Sidereal Time.

1. Introduction

Torques on the oblate Earth due to the gravitational attraction of the Sun and Moon cause the Earth's equator to precess and nutate. The precession is retrograde and its rate is 50 "/yr, roughly 1/3 of it due to the Sun and 2/3 from the Moon, The rate depends on the lunar and solar masses and distances, the orbital eccentricities and inclinations, and the obliquity angle between the Earth's equator and ecliptic planes,

Recent decades have seen impressive advances in the accuracies of techniques measuring positions of artificial satellites, the Moon, and radio sources. Accurate, theories for the motion of the Earth's equator in space are needed. This paper examines several theoretical contributions to precession, obliquity change, and nutation.

The orbit of the Earth-Moon system about the Sun defines the ecliptic plane. The lunar orbit is inclined 5° to the ecliptic plane and the strong solar torques drive the precession of the lunar orbit plane along the ecliptic with an 18.6-yr period. But several influences cause a slight tilt of the mean plane of orbital precession with respect to the ecliptic. The Earth's oblateness contributes a small torque which attempts to precess the lunar orbit along the equator, The net result of these two torques is a lunar orbit precession along a plane tilted 8" with respect to the ecliptic and this plane intersects the ecliptic at the dynamical equinox, the intersection of the ecliptic and equator planes. This small influence of the Earth's oblateness on the lunar orbit in turn causes a small change in the precession of the Earth's equator.

The orbit planes of the planets have small inclinations with respect to the ecliptic plane. As a consequence of the planetary attractions, the ecliptic plane moves. The Moon's mean plane of orbital precession follows the moving ecliptic closely, but not perfectly, "l'his motion causes a 1,4" tilt of the plane of orbital precession to the ecliptic. l'here arc. also direct planetary torques on the lunar orbit which contribute a smaller displacement. These two influences on the lunar orbit cause torques on the oblate Earth which modify its orientation, as does the direct attraction of the planets. The torques from these three planetary influences are not aligned with respect to the dynamical equinox. Consequently, they contribute to both the precession of the equator and the obliquity rate, While the precession rate must be a measured quantity, the obliquity rate is not a free parameter of the dynamics. These planetary influences are not included in the IAU-adopted theory of precession and obliquity change (I .ieske *et al.* 1977). Neither have all of the consequences of the planetary tilts on the lunar orbit been included in recent nutation theories.

The above outlined corrections to the motion of the Earth's equator are developed in the following sections. To these corrections are added precession corrections developed by Kinoshita & Souchay (1990) from the Earth's J4 and second-order corrections due to nutations. From the revised theory are developed new expressions for the motion of the Earth's equator and revised values of the Earth's fractional moment of inertia, $(C-A)/C$, and the normalized polar moment, C/MR^2.

2. Fundamentals

This section will set up the fundamental equations for calculating the motion of the Earth's equator (or pole) in space. As the computations of the subsequent two sections are limited to small effects, it is reasonable to introduce simplifications. The Earth will be treated as a rigid body without oceans and without the influence of a liquid core. Small differences in the directions of the axes of angular momentum, instantaneous spin, and figure (equivalent to celestial ephemeris pole for a rigid body) are ignored. The equations will be written for the angular momentum axis, but strictly speaking it is the motion of the figure axis of the rigid Earth which is desired, Also ignored are second-order effects due to the change of the Earth's orientation, e. g., precession and nutation modifying the computation of precession and nutation.

The oblate, rigid Earth is torqued by an external body. The attracting body has a geocentric distance r and a product of the gravitational constant and mass Gm. The Earth has moments of inertia A, A, C with A<C and mass M. The z axis is aligned with the Barth's principal axis corresponding to the maximum moment C, and the x axis points toward the intersection of the ecliptic and equator planes, the dynamical equinox. The potential energy of the external body in the gravity field of the oblate Earth is

$$V = Gm \, [\, M / r - (C-A)(\, 3 \sin^2\delta - 1 \,) / 2 \, r^3 \,] \qquad (1)$$

where the declination of the attracting body is δ and the right ascension is α Equivalently, the vector $\underline{r}$ has components (x, y, z). The torque $\underline{T}$ on the Earth is

$$\underline{T} = \quad - \underline{r} \times \underline{\nabla} V \qquad (2)$$

$$\underline{T} = \cdot \, - \cdot \, - \cdot \cdot \frac{3 \; G \; m \; (C-A) \sin \delta \cos \delta}{r^3} \begin{bmatrix} \sin \alpha \\ -\cos a \\ 0 \end{bmatrix}$$

$$T = \frac{3 G m (C-A)}{r^5} \begin{pmatrix} yy\,z \\ -x\,z \\ 0 \end{pmatrix}$$ (3)

The rate of change of the vector angular momentum L is governed by $dL / dt = T$. Given the orbit of the external attracting body, the resulting precession and nutation of the Earth can be calculated.

The analytical theories for the Sun, planets, and Moon are referred to the ecliptic plane. Consequently, the conversion from geocentric ecliptic coordinates (X, Y, Z) to equatorial coordinates (x, y, z) requires a rotation about the x axis by the obliquity ε.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X \\ Y \cos \varepsilon - Z \sin \varepsilon \\ Z \cos \varepsilon + Y \sin \varepsilon \end{pmatrix}$$ (4)

In the torque vector the products of equatorial coordinate components become

$$\begin{pmatrix} yz \\ -xz \\ 0 \end{pmatrix} = \begin{pmatrix} (Y^2-Z^2) \sin 2\varepsilon / 2 + YZ \cos 2\varepsilon \\ - XZ \cos \varepsilon - XY \sin \varepsilon \\ 0 \end{pmatrix}$$ (5)

The ecliptic coordinates (X, Y, Z) of the attracting body can be written in terms of the geocentric distance r and the geocentric ecliptic longitude λ and latitude β.

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = r \begin{pmatrix} \cos \beta \cos \lambda \\ \cos \beta \sin \lambda \\ \sin \beta \end{pmatrix}$$ (6)

Because the Earth's path about the Sun is well approximated by an elliptical orbit in the ecliptic plane, the solar torque may be computed with good accuracy with little effort. Averaged over an integral number of revolutions the average x-component of torque is

$$T_x = 3 G m (C-A) \sin \varepsilon \cos \varepsilon / 2 a^3 (1 - e^2)^{3/2}$$

where a is the semimajor axis, e the orbital eccentricity, and ε is the obliquity. The x-component of torque gives rise to a retrograde precession along the ecliptic with rate $d\psi/dt = T_x/ C\omega_z \sin \varepsilon$ where ω_z is the major component of the Earth's angular velocity and $C\omega_z$ approximates the total angular momentum of the Earth's spin.

$$d\psi/dt = 3 G m (C-A) \cos \varepsilon / 2 a^3 (1- e^2)3/2 \, C\omega$$ (7)

G/a^3 maybe replaced with n^2 divided by the sum of the masses (Sun+Earth+Moon) using Kepler's third law. The analogous precession from the Moon includes an inclination factor of $1 - 1.5 \sin^2 i$. The other two torque components have zero average,

but of course the first two components have time variations which contribute periodic nutation terms.

The elliptical approximation above works well for the solar-induced precession of the Earth's equator along the ecliptic, but it is a coarser approximation for the lunar effect because the lunar orbit is strongly perturbed by the Sun. These difficulties in the major precession and nutation effects have been dealt with by Kinoshita & Souchay (1990), Their computation for the solar precession is only larger by $2x 10^{-6}$ so Eq. (7) is a very good approximation for the Sun. The lunar orbit is highly perturbed and the equivalent equation for the lunar-induced precession, including the inclination factor, is less precise. The computation of many small corrections in this paper can use the elliptical approximation

3. Effects Due to the Tilted Lunar Mean Plane

The lunar orbit precesses along a plane which is tilted slightly with respect to the ecliptic, The Earth's J2 causes an 8" tilt and planetary effects cause a 1.5" tilt. As a consequence of these small sizes, expansions will be used. The lunar latitude arises from the 5.15° inclination of the orbit to the ecliptic i, and smaller perturbations $\Delta\beta$ so that

$$\sin\ \beta \approx\ \sin i \sin F - t\ \Delta\beta \tag{8}$$

where F is the mean argument of latitude, Similarly the lunar longitude λ may be written in terms of its mean longitude 1.., mean anomaly ℓ, eccentricity e, and smaller contributions AL

$$\lambda\ =\ L\ +\ 2 e \sin\ell\ +\ \Delta L \tag{9}$$

The perturbing terms most important for precession and nutations are selected from Chapront-Touze & Chapront (1988, 1991). For .12 perturbations

$$\Delta\beta = -8.045\text{"} \sin L + 0.326\text{"} \sin(L\text{-}2F)$$

$$AL =\ \ 7.063\text{"} \sin \Omega + 0.361\text{"} \sin(L\text{+}F) \tag{1o}$$

where Q is the lunar node $(\Omega{=}L{-}F)$. The important planetary-induced terms are

$$\Delta\beta =\ \ 1.510\text{"} \sin(L{+}96.68°)$$

$$AL = -0.289\text{"} \sin(\Omega{+}95.13°)\ \ -0.062\text{"} \sin(L{+}F{+}95.13°) \tag{11}$$

The J2 and planetary effects also cause radial perturbations, but compared with the longitude and latitude perturbations they are relatively ineffective in modifying precession and nutations.

For the purposes of expansions, the above perturbations in ecliptic longitude and latitude will be represented symbolically as

$$\Delta\beta =\ B\ \sin(L{+}\phi) + B'\ \sin(L\text{-}2F)$$

$$AL =\ E\ \sin(\Omega{+}\varphi)\ +\ E'\ \sin(L{+}F{+}\varphi) \tag{12}$$

The ecliptic plane is rotating about a line which is displaced from the dynamical equinox by φ–90°. The phase ϕ is different from φ because the first term in latitude combines both the direct effect of the planets with the indirect effect of the ecliptic motion. After introducing these perturbations into the differential equations of the previous section and carrying out the expansions through first degree in e and sin i there arc contributions to both rate and periodic terms in ψ and ε. The rate terms are

$$d\psi/dt = \frac{3Gm\,(C–A)\cos 2\varepsilon}{2\,a^3\,C\,\omega_z\,\sin\varepsilon}\;[\,B\cos\phi + (E' – E)\sin i\,\cos\varphi/2\,]$$

$$d\varepsilon/dt = \frac{–3Gm\,(C\text{-}A)\,COS\,\varepsilon}{2\,a^3\,C\,\omega_z}\;[\,B\sin\phi + (E' – E)\sin i\,\sin\varphi/2\,] \qquad (13)$$

A contribution to the obliquity rate requires phase shifts. The planetary effects contribute a --0,254 mas/yr (mas=milliarcsecond) correction to the obliquity rate while J2 perturbations contribute nothing. To the precession, J2 perturbations contribute -2.630 mas/yr while planetary effects contribute -0.056 mas/yr for a total of -2,686 mas/yr.

The largest of the nutation corrections has the 18.6-yr period of the lunar node. While the rigid-body nutation theory caused by the main lunar theory only contains in-phase terms (sines of the arguments for $\Delta\psi$ and cosines for $\Delta\varepsilon$), the phase shifts from the planetary effects also induce out-of-phase terms (cosines for $\Delta\psi$ and sines for $\Delta\varepsilon$).

$$\sin\varepsilon\,\Delta\psi = \frac{3Gm\,(C–A)\sin E\cos\varepsilon}{2\,a^3\,C\,\omega_z\,d\Omega/dt}\;\{\,[--(7/2)\sin i\,B\cos\phi + E'\cos\varphi + 3\sin i\,B']\sin\Omega$$

$$+\;[(5/2)\sin i\,B\sin\phi – E'\sin\varphi]\cos\Omega\,\}$$

$$\Delta\varepsilon = \frac{3Gm\,(C–A)\sin\varepsilon}{2\,a^3\,C\,\omega_z\,d\Omega/dt}\;(\,[\,\sin i\,B\cos\phi/2 -- E'\cos\varphi]\cos Q$$

$$+\;[\sin i\,B\sin\phi/2 – E'\sin\varphi]\sin\Omega\,\} \qquad (14)$$

Nutation terms at half of the nodal period must also be considered, The contributions to the nutation terms with argument twice the lunar node are

$$\sin\varepsilon\,\Delta\psi = \frac{3\,G\,m\,(C–A)\cos 2\varepsilon}{4\,a^3\,C\,\omega_z\,d\Omega/dt}\;[\,(\sin i\,E\cos\varphi/2\text{-}B')\sin 2\Omega$$

$$+\;(\sin i\,E\sin\varphi/2)\cos 2\Omega\,]$$

$$\Delta\varepsilon = –\frac{3Gm\,(C–A)\cos\varepsilon}{4\,a3\,C\,\omega_z\,d\Omega/dt}\;[\,-(\sin i\,E\cos\varphi/2\text{-}B')\cos 2\Omega$$

$$+\;(\sin i\,E\sin\varphi/2)\sin 2\Omega\,] \qquad (15)$$

Finally, there are small corrections to terms with argument 2L.

$$\sin \varepsilon \, \Delta\psi = \frac{3\,G\,m\,(C-A)\,COS\,2F.}{4\,a^3\,C\,\omega_L\,dL/dt} \; [-\,B\cos\phi\,\sin\,2L.\;-\;B\sin\phi\cos\,2L\;]$$

$$\Delta\varepsilon = \frac{3Gm\,(C-A)\,\cos\varepsilon}{4\,a^3\,C\,\omega_L\,dL/dt} \; [\,B\,\cos\,\phi\cos\,2L,\;-\,B\sin\phi\,\sin\,2L.\;] \qquad (16)$$

Using the numerical values of the coefficients and phases for the J2 and planetary effects, the above contributions to the nutations have been calculated. They are presented in Table 1 (units mas). The major contribution is to the 18.6-yr $\Delta\psi$ term with a lesser contribution to the 18.6-yr $\Delta\varepsilon$ term. Both of these contributions increase the magnitude of the conventional 18.6-yr terms, The contributions to the 9.3-yr nutations are small. The two contributions in Eq. (15) from the J2 effect nearly cancel and the values in the table are effectively zero. The higher frequency of the 2L (=2F+2Ω) term prevents those half-month nutation corrections from being large.

Out-of-phase terms in nutation theory will arise from dissipative processes in the oceans (Wahr & Sasao 1981, Zhu et al. 1990) and interior of the Earth (Wahr & Bergen 1986, Dehant 1988, 1990). The out-of-phase terms in Table 1 arise from the phase shifts in the planetary effects which in turn arise because the orbit planes of the planets other than Earth have no special alignment with the ecliptic plane or dynamical equinox. There are still smaller corrections at arguments of 2L + Ω, 2L - Ω, 2ϖ+Ω, 2ϖ-Ω, and 3Ω which are not given.

Woolard (1953) was aware that out-of-phase terms in nutation theory could arise from planetary perturbations on the lunar orbit, The out-of-phase 18.6-yr term for nutation in longitude occurs in his Table 24 (it is marked with a ? and a footnote, but matches the value in Table 1 of this paper), but the obliquity term was too small for his cutoff limit. In the same table (nutations with respect to space) Woolard shows additional 18.6-yr out-of-phase terms arising from the integration oft sin Ω and t cos Ω terms in the differential equations, but they do not show up in Table 26 (nutations with respect to moving ecliptic) though one is big enough, Woolard also calculated the obliquity rate contribution, showing -0.256 mas/yr in his Table 24 at the year 1900. In the text (pg. 127) he also comments that the planetary-induced lunar terms contribute to precession and to the acceleration of obliquity. Kinoshita (1975, 1977) considered the obliquity rate contribution to be due to an error in Woolard's equations of motion. This assertion will be discussed further in section 6. Kinoshita's M1 correction to precession is --2.68 mas/yr and it appears to correspond to the sum of the J2 and in-phase planetary effects computed in this paper. In Kinoshita & Souchay (1990) a more elaborate "second-order" correction to precession replaces the earlier M1 correction. It contains a -2.60 mas/yr correction to precession due to the J2 effects, but the -0.056 mas/yr planetary effect is missing. Presumably, their nutations contain the corresponding contribution from J2, but not from the planetary tilt.

4., Rates Due to Direct Torques of Planets on Earth

The torques from the Sun and Moon dominate the precession of the Earth's equator. "1'here arc small additional torques from the planets which contribute to the precession. The inclination of the planetary orbits to the ecliptic will also cause a small obliquity rate. A calculation of the precession was given by Kinoshita & Souchay (1990), but not the obliquity rate. A brief derivation of both rates is given below.

in order to compute the geocentric coordinates of the attracting planet, it is necessary to difference the heliocentric coordinates of the planet and the Earth. Primes will be used for the planet's variables, no primes for the Earth. The effects are small so to keep the derivation from becoming unwieldy two approximations will be introduced. The heliocentric orbits will be taken as circles and the planetary inclinations will only be carried to first degree ($\cos i' \approx 1$). Then the planet's geocentric ecliptic coordinates are

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} a'\cos(u'+\Omega') - a\cos(u+\Omega') \\ a'\sin(u'+\Omega') - a\sin(u+\Omega') \\ a'\sin i'\sin u' \end{pmatrix} \qquad (17)$$

where a and a' denote the semimajor axes, i' the inclination to the ecliptic, Ω' the node on the ecliptic, and u and u' the arguments of latitude measured from the same node for both the Earth and attracting planet. The geocentric distance r is given by

$$r^2 = a^2 + a'^2 - 2aa'\cos(u-u') \qquad (18)$$

The ecliptic coordinates are rotated into equatorial coordinates following Eq. (4) and the products of coordinates of Eq. (5) are formed for substitution into Eq. (3) for the torque. There results expressions involving products of sines and cosines of u and u'. In order to isolate the secular rates from the periodic terms Gauss' method of averaging over u and u' is used. Denoting the average with <>, an example is $<(YZ/r^5)> = \iint (YZ/r^5)\,du\,du'/4\pi^2$ with both integrals evaluated from O to 2π. To winnow out terms which will disappear during the double integration a mathematical device is useful. The transformations (sin u, sin u')$\rightarrow$(–sin u, -sin u'), (cos u, cos u')$\rightarrow$(–cos u, –cos u'), and both taken together leave unchanged the distance r which appears in the denominators of the integrals. Any component of the numerators $Y^2-Z^2, YZ, X2$, or XY which reverses sign under any of the three transformations will average to zero. Also, since only u–u' appears in the denominator, changing variables of integration to u-u' and u+u' makes it clear that additional components average to zero. Finally one gets

$$<(Y^2-Z^2)/r^5> = <1/r^3>/2$$
$$<YZ/r^5> = (a'/2)\sin i'\cos\Omega'\,(a'<1/r^5> - a<\cos(u-u')/r^5>)$$
$$<XZ/r^5> = -(a'/2)\sin i'\sin\Omega'\,(a'<1/r^5> - a<\cos(u-u')/r^5>)$$
$$<XY/r^5> = 0 \qquad (19)$$

The three different averages on the right-hand sides above are only functions of u--u' and they may be evaluated in terms of complete elliptic integrals of the first and second kind, K(k) and E(k), respectively,

$$<1/r^3> = 2E(k)/[\pi(a+a')^3(1-k^2)]$$
$$<1/r^5> = 2[-K(k)+2E(k)(2-k^2)/(1-k^2)]/[3\pi(a+a')^5(1-k^2)]$$
$$<\cos(u-u')/r^5> = 2[-K(k)(2-k^2)+2E(k)(1-k^2+k^4)/(1-k^2)]/[3\pi(a+a')^5 k2(1-k^2)]$$

The modulus k is the geometric mean of the two semimajor axes divided by the arithmetic mean or

$$k^2 = 4\ a\ a'\ /\ (\ a + a'\ )^2 \qquad (21)$$

The rates induced by the direct planetary torques are

$$d\psi/dt = [Gin'\ (C-A)/\ \pi\ (a+a')(a-a')^2\ C\omega_z]\ [3\ \cos\ \varepsilon\ E(k) - G\ \cos\ 2\varepsilon\ \sin\ i'\ \cos\ \Omega'\ /\ \sin\ \varepsilon\ ]$$
$$d\varepsilon/dt = [Gin'\ (C--A)/\ \pi\ (a+a')(a-a')^2\ C\omega_z]\ [\ -\ G\ \cos\ \varepsilon\ \sin\ i'\ \sin\ \Omega'\ ]$$
$$G = (\ a^2 + 7\ a'^2\ )\ E(k)\ /\ 2\ (\ a^2 - a'^2\ )-(\ a-a'\ )\ K(k)\ /\ 2\ (a+a') \qquad (22)$$

where m' is the attracting planet's mass. In the precession rate the term involving E(k) allows for the larger contribution to precession as though the attracting body were in the ecliptic plane. For both rates the combinations sin i' sin Ω' and sin i' cos Ω', which are two of the coordinates of the planet's orbit pole direction, allow for small contributions due to the tilt of the planet's orbit plane with respect to the ecliptic. .

The numerical results for the precession and obliquity rate contributions from the direct planetary torques on the Earth are given in Table 2. The 0.3183 mas/yr precession rate results from 0.3269 mas/yr due to the E(k) term and -0.0086 mas/yr from the planetary inclinations. The comparison of precession rate with Kinoshita & Souchay's (1990) computations for Venus through Saturn shows a 3% difference for Venus and 1 % for Jupiter. In Table 2 the largest values of the modulus k occur for Venus and Mars, 0.987 and 0.978, respectively, The obliquity rate contribution of-0.014 mas/yr combines with the larger contribution of –0.254 mas/yr from planetary effects through the lunar orbit (section 3) to give -0.268 mas/yr. Tidal torques contribute an additional 0.024 mas/yr to obliquity rate; the derivation is interconnected with nonlinear contributions and will be deferred (section 7) until after the summarizing of the rates, The total obliquity rate with respect to space is –0.244 mas/yr. This correction to the obliquity rate is not included in the expressions accompanying the IAU-adopted precession theory.

5. Total Precession and Obliquity Rates

This section summarizes the various contributions to precession and obliquity rates, gives the total values, and discusses the implications. The precession and nutations of the Earth's pole in space depend on the dynamical flattening (C–A)/C. Since the precession can be measured with a smaller relative error than the nutations, the value of precession is a primary IAU-adopted constant and recent nutation series have been computed from the derived value of (C–A)/C (or proportional quantities for the Sun and Moon called ks and k_M).

Knowledge of the precession rate and obliquity has improved since the adoption of the 1976 IAU constants. The value of (C–A)/C appropriate to the IAU constants, but with the theoretical modifications of this paper and updated ecliptic motion, is 0.00327397826, The featured computations will use improved values of the precession rate, obliquity, masses, mean motions, and ecliptic motion. A –3 mas/yr correction to the IAU-adopted value of the precession constant has been indicated by several lines of evidence: lunar laser ranging (Williams *et al.* 1991, 1993), very long baseline interferometry (Herring *et al.* 1991, Herring 1991, McCarthy & Luzum 1991, Steppe *et al.* 1993), the two combined (Chariot *et al.* 1991), and systematic proper motions in star catalogs (Miyamoto & Soma

1993). Several recent fits have given corrections near -3.2 to –3.3 mas/yr and a general precession rate of 5028.77 "/cy has been chosen. The change from the IAU general precession rate is –3.266 mas/yr and the change in the luni-solar precession rate is -3.219 mas/yr (the two do not match because the ecliptic motion is different from the IAU paper). For the obliquity at J2000, the value of 84381.409" = 23°26'21.409" is based on analyses of lunar and planetary observations. This obliquity and the mass ratios Earth/Moon = 81.30059 and Sun/(Earth+Moon) = 328900.560 are from the recent ephemeris DE 245 (Newhall, Standish, & Williams 1993). See Standish (1982) for the technique of extracting the obliquity from an ephemeris. The corresponding (C- A)/C is 0.00327376340, ks = 3475,19739 "/cy (cy=century), and k_M = 7546,73700 "/cy (or 7567.30S75 "/cy with the $1/F_2{}^3$ factor).

The various contributors to precession and obliquity rates are. summarized in Table 3. Taken from Kinoshita & Souchay (1990) are the first-order equations for the computation of the lunar- and solar-induced precession (the values were computed from the equations), the value of the second-order lunar plus solar effects (excluding the J2 orbit effects), and the small value for the precession induced by the Earth's J4 gravitational harmonic, The contribution to precession and obliquity rate due to the lunar orbit tilt comes from section 3 of this paper and the planetary contribution due to direct torques on the Earth's oblateness comes from Table 2 in section 4. The relativistic precession, variously called the geodesic, geodetic, and de Sitter-Folker precession, is computed from the following equation based on that in Barker & O'Connell (1970, 1975).

$$P_g = 3 \ (n \ a \ / \ c)^2 \ n \ / \ 2 \ (1 - e^2) \tag{23}$$

c is the speed of light and n and a are the mean motion and semimajor axis of the orbit of the Earth-Moon system about the Sun. 'l'he convention of measuring the precession constant in a left-handed sense (retrograde) results in a negative sign for the geodesic precession in the table. The tidal influence on obliquity rate is taken from section 7. The sum of all of the above contributions gives the precession and obliquity rate with respect to space for the stated value of (C–A)/C. Conventionally the precession along the fixed ecliptic with respect to space is referred to as the "luni-solar precession" (which includes contributions from the planets as well). Clearly, it would be inappropriate to refer to the companion -0.244 mas/yr obliquity rate as luni-solar obliquity rate since most of it ultimately comes from planetary influences.

To get the precession and obliquity rate for the moving equator with respect to the moving ecliptic plane it is necessary to subtract off the motion of the ecliptic plane. This is done in the last two lines of Table 3. The values for the ecliptic motion have been improved upon since the IAU theory (Lieske *et al.* 1977). improved ecliptic motion and its influence on the precession expressions has been considered by Bretagnon (1982), Bretagnon & Chapront (1981), Laskar (1986), and Simon *et al. (1993)* and the improved motion from Simon *et al.*, including the correction for mass changes, has been used in the table, There is a problem with the nomenclature of the past. What have been called "planetary precession" and obliquity rate (Woolard uses "precession in obliquity") come from the motion of the ecliptic. We now have two planetary contributions to each of precession and obliquity rate which are motions in space, not ecliptic motion. It is conventional to refer to the (mean-of-date) motion of the dynamical equinox along the moving ecliptic plane as "general precession in longitude". Consequently, the final line has been labeled general motion and by extension the obliquity rate might be called general obliquity rate.

Both very long baseline interferometry (VLBI) and lunar laser ranging, (LLR) are capable of measuring the motion of the equator with respect to space rather than the moving ecliptic. Thus both measure the luni-solar precession rate, not the general precession rate (the IAU primary constant), and have the potential to measure the --0.244 mas/yr obliquity rate with respect to space. There is weak evidence for the latter in the VLBI results (Herring *et al*. 1991, Steppe *et al*. 1993). Better precession and obliquity rate measurements can be anticipated as the VLBI and LLR data spans lengthen and separation of rates and 18.6 yr nutations becomes stronger.

The nutation theory of Kinoshita & Souchay (1990) is a significant improvement on previous theories and it is important to understand the corresponding values of (C-A) /C, k_M, and ks. Unfortunately, three different values of (C–A)/C and two of k_M have been published and it is important resolve the discrepancy. From the 1976 IAU constants and Kinoshita and Souchay's numerical values and equations is calculated (C–A)/C = 0.00327396771, k_S = 3475.41426 "/cy, k_M = 7547.19969 "/cy (or 7567.76970 "/cy with the $1/F_2^3$ factor), The set of their values which have internal consistency is (C–A)/C = 0.003273967 (Souchay & Kinoshita 1991), ks = 3475.4135 "/cy, and k_M = 7547.1981 "/cy (7567,7681 "/cy with factor), The unexplained relative difference is 2.2x 10-7, but is small enough (o only influence the nutations at the few microarcsecond level. In Kinoshita and Souchay the (Earth+Moon)/Sun mass ratio is incorrectly labeled Earth/Sun ratio, but this discrepancy is too small to explain the difference. To adjust their theory to the (C–A)/C and precession rate of this paper with all of the corrections, multiply their nutation series by the factor 0.99993782.

It is instructive to consider several contributions to the above factor and the proportionate (C-A) /C, The largest is the correction to the IAU precession rate causing a relative change of –6.48x 10–5. Updating the mass ratios and obliquity causes -8x 10^{-7}. The change in the ecliptic motion causes 9x 1 ()⁻7. Theoretical differences due mainly to the planetary tilt induced precession (absent in Kinoshita and Souchay) and a somewhat different geodesic precession account for 2.2x10⁻6.

The ratio of J_2 = (C–A)/MR^2, where R is the Earth's equatorial radius, and (C--A)/C gives C/MR2 the normalized polar moment of inertia, Combining J_2 from the GEM-T2 solution of Marsh *et al. (1990)*, including a suitable addition for the model's permanent tide, with the precession derived (C–A)/C from above yields a rigid body C/MR^2 = 0.3307007. With the mean moment I = (C+2A)/3, then I/MR^2 = 0.3299789,

6. The Phase of the Tilt Terms

There are two reasons to consider the seemingly prosaic subject of the phase of the tilt terms in section 3. 1) It has been stated (Kinoshita 1975, 1977) that these terms do not give rise to an obliquity rate and the resolution of the difference between that claim and this paper (and Woolard 1953) hinges on the origin of the phase. 2) Time variations of the phase will give rise to higher derivatives of the precession and obliquity.

The J_2-induced tilt terms in section 3 have zero phase so long as L and Ω are referred to the moving equinox. They do not give rise to an obliquity rate and do not need to be considered further in this section. The planetary-induced tilt terms in the lunar orbit arise in two ways. The direct terms arise from the forces of the planets on the lunar orbit. The indirect terms arise from the force of the Sun on the lunar orbit, coupled with the motion of the ecliptic plane due to the forces of the planets changing the heliocentric orbit. The

tilt from the direct effect is an order-of-magnitude smaller than the indirect effect. The two components have been combined in section 3. The indirect contribution will dominate the following discussion.

We wish to express the secular motion of the lunar orbit plane acted upon by the Sun, The coordinates of the pole of the variable lunar orbit are $(P_v, -Q_v, \cos iv)$ where $P_v = \sin iv \sin \Omega v$ and $Q_v = \sin iv \cos \Omega v$. The analogous variables for the ecliptic pole are P' and Q'. Using an inertial frame aligned with the ecliptic and equinox at the initial time, e. g. at J2000 $P' = Q' = O$, a good approximation for the differential equations for the. secular motion is (see chapters 12 and 16 of Brouwer & Clemence 1961)

$$dP_v/dt = \dot{\Omega} (Q_v - Q')$$
$$dQ_v/dt = -\dot{\Omega} (P_v - P') \tag{24}$$

In the first approximation $\dot{\Omega}$ is a quantity which is proportional to the mass ratio Sun/(Earth+ Moon), the lunar mean motion, and the cube of the ratio of the lunar to heliocentric semimajor axes $(a/a')^3$. When the ratio of ecliptic motion to $\dot{\Omega}$ is small, it is 10–5 for the Moon, a good approximation for the solution of the differential equations is

$$P_v = P + P' - \dot{Q}'/\dot{\Omega}$$
$$Q_v = Q + Q' + \dot{P}'/\dot{\Omega} \tag{25}$$

where $P = \sin i \sin \Omega$ and $Q = \sin i \cos \Omega$ represent a uniformly processing lunar orbit plane with rate $\dot{\Omega}$ (retrograde 18.6 yr period) and fixed inclination i. P_v -P' and Q_v – Q' are good approximations for the motion of the lunar orbit pole with respect to the moving ecliptic pole. In the above solution the orbit is precessing along a plane tilted slightly with respect to the moving ecliptic. At J2000 $\dot{Q}'/\dot{\Omega}$ = 1.386" and $\dot{P}'/\dot{\Omega}$ = -0.124", so the indirect term causes a 1.39" tilt with a phase governed by the node of rotation of the ecliptic plane (Π = 174.87° at J2000). In Eq. (11) the phase of the indirect contribution is 270°- 11= 9S. 13" at J2000 and the arguments depend on L - Π and Ω - Π. Conventionally L and Ω are measured from the moving (mean of date) equinox along the moving ecliptic so for compatibility Π must be measured from the moving equinox, along the moving ecliptic, to the node of rotation of the ecliptic on the moving ecliptic (in the notation of the IAU precession paper this is Π(T,0)). The smaller direct contribution has its own phase which depends on the planetary nodes, so the phase of the combined direct and indirect terms is slightly larger.

Woolard (1953) earlier computed the obliquity rate contribution from the planetary-induced tilt in the lunar orbit, When explaining it (pgs. 127-128) he broke the $\Delta\beta$ contribution of Eq. (11) into sine and cosine components of $F + \Omega = I.$, displaying only the cosine component, presumably because he knew that the sine does not give rise to a secular rate. If the argument was measured with respect to the moving equinox, as Ω and L conventionally are in lunar theory, that is if there was no Π, then there would be no obliquity rate. For this reason Kinoshita (1975, 1977) argued against Woolard's obliquity rate, but the discussion in the appendix of the latter paper seems to be unaware that there was a phase, undisplayed in Woolard, arising from an external source. It is concluded here that the obliquity rate from the indirect tilt is real. In addition, the phase of the direct tilt and the direct action of the planets on the Earth's bulge both depend on the planetary nodes so those obliquity rates are also valid.

There is an anomaly that I do not understand, The comparison by Souchay & Kinoshita (1991) of their theory with a numerical integration showed as discrepancies neither the obliquity rate term nor the out-of-phase 18.6-yr nutation terms which arise from the same source.

When measured from the moving equinox 11 increases and the phase decreases by 0.915 O/century (this rate is wrong in Brown's lunar theory) and the precession and obliquity rates arising from the indirect tilt will not be constant. The time dependence of the P' and Q' derivatives in Eq. (25) above can be inserted in Eqs. (12) and (13) to compute the precession and obliquity rates as a power series in time. For use in section 8 it is convenient to express these rates in a coordinate system moving with the ecliptic and equinox. In the two-time (T, t) power series of Lieske *et al.* (1977) or Simon *et al.* (1993) differentiate P' and Q' with respect to t, set t = O, and use T as the time variable. The series for the indirect contribution then becomes $-0.00392 + 0.000703$ T for precession rate (units " and centuries=cy) and $-0.0233 + 3 \times 10\text{-}6$ T^2 for obliquity rate. The computation of the nonlinear contributions is similar for the direct planetary torques on the Earth and the direct planetary tilt on the lunar orbit. In Eqs. (22) the dependence of the direct torque effects on the planetary Ps and Qs is explicit and power series for the Ps and Qs (Laskar 1986, Simon *et al.* 1993) can be used. Venus dominates the acceleration and the result is $-17x$ $10\text{-}6$ $"/cy^2$ for precession and $2x$ 10^{-6} $"/cy^2$ for obliquity. The acceleration due to the direct tilt terms is more difficult, but is estimated to be about 40% larger than for the direct torque effect. Because the polynomials for the planets used a fixed equinox, to these figures must be added the larger accelerations which result from transforming from a fixed to a moving equinox: $411 \times 10\text{-}6$ $"/cy^2$ for precession and -48×10^{-6} $"/cy^2$ for obliquity, The total of the preceding direct and indirect tilt terms and the direct torques on Earth are listed under "planetary tilt and direct torque" in Table 4. Note that part (0,03269 "/cy) of the direct torque effect for precession in Eq. (22) does not depend on planetary Ps and Qs, contributes no accelerations, and is included with the entry for luni-solar precession in the table,

The coefficients of the planetary-tilt-induced nutations (Table 3) will also have secular changes. Assuming that the secular changes in the ΔL coefficients scale in proportion to those of the latitude coefficient, then the in-phase contributions are $(-0.0301 + 0.0050T) \sin \Omega$ to the longitude nutation and $(0.0029 - 0.0005T) \cos \Omega$ to the latitude nutation (units mas and cy). The relative changes of the out-of-phase coefficients is about 10^{-3} /cy and is ignorable.

7. Tidal and Nonlinear Effects

This section will consider effects which influence the accumulated precession accelerations (first derivative of precession rate), higher derivatives, and another contribution to obliquity rate. There are effects due to the change in the eccentricity of the orbit of the Earth-Moon system about the Sun which have been considered in previous theories, plus tidal effects in the lunar orbit and Earth rotation, and possible changes in (C-A)/C. Many of the results of this section can be derived from Eq. (7) and its lunar counterpart. Moving toward polynomial expressions for the precession quantities as in Lieske *et al. (1977),* the units of that paper are adopted now (arcseconds and centuries = cy). The results of this section arc summarized in Table 4. That table also gives the ε dependence since the change of obliquity contributes additional accelerations and higher derivatives which will be utilized in the next section.

As seen from Eq. (7), the eccentricity of the heliocentric orbit enters into the solar-induced precession of the equator and changes in the eccentricity will affect the derivatives of the precession rate. The evaluation of the eccentricity-induced acceleration (first derivative of the precession rate) in the accumulated precession in the IAU theory dates to de Sitter & Brouwer (1938). It is reevaluated here at J2000. Using the eccentricity polynomials in Laskar (1986) or Sitnon *et al.* (1993), $de/dt = -42.0 \times 10^{-6}$ /cy (cy=century). This causes the solar-induced precession rate to have a first derivative of -3362×10^{-6} "/cy^2. The eccentricity of the heliocentric orbit also enters into the geodesic precession Eq. (23). The derivative of that rate (in retrograde sense) is 2.7×10^{-6} "/cy^2. The lunar orbit includes perturbations by the Sun and some of these depend on the heliocentric eccentricity. These are small influences on the lunar-induced precession which enter as the squares of periodic terms in the lunar latitude and distance. The fractional influence on the precession rate by the radial terms is 1.97×10^{-6} and by the latitude terms is -5×10^{-8} yielding -33.3×10^{-6} "/cy^2 or 1.0% of the solar-induced part. The luni-solar acceleration is -3395×10^{-6} "/cy^2 exclusive of the contribution of the geodesic precession. There is also a small contribution due to the second derivative of e2. The t and t2 terms in the precession rate (without geodesic precession) due to heliocentric eccentricity changes are -3395×10^{-6} "/cy^2 t $- 6 \times 10^{-6}$ "/cy^{3}t^2. To convert the t coefficient to the P_1 parameter of the IAU theory divide by cos ε_0 to get -0.00370 "/cy^2 versus -0.00369 "/cy^2 used for the IAU theory . The agreement is excellent, aided by the small t^2 term.

Tides are raised on the Earth by the Moon and Sun and their energy dissipation causes the Moon to recede and the Earth's rotation to slow. Lunar laser analyses indicate a secular acceleration of -26.0 "/cy^2 and a tidal (semi major axis) recession of 3.84 cm/yr (Williams *et al.* 1993) so $da/dt / a = 1.00 \times 10^{-8}$ /cy. The $1/a^3$ dependence of precession in Eq. (7) implies the precession changes by $-) 03 \times 10^{-6}$ "/cy^2. The tidal changes in the lunar orbit eccentricity e and inclination i are small (Chapront-Touze & Chapront 1983, 1988) and lead to only 0.9×10^{-6} "/cy^2 and 0.2×10^{-6} "/cy^2, respectively, in the precession. Eq. (7) depends on the obliquity ε, the angular momentum $C\omega_z$, and the moment difference (C-A) which exhibit secular changes due to tidal effects. An angular momentum balance between the Earth's spin and the lunar orbit for long time scales gives an estimate for these secular changes. Writing the angular momentum components perpendicular and parallel to the ecliptic plane

$$H_z = C\omega_z \cos \varepsilon + Mm \, [G(M+m) \, a \, (1-e^2)]^{1/2} \cos i / (M+m)$$

$$H_y = C\omega_z \sin \varepsilon \tag{26}$$

where m and M are the masses of the Moon and Earth and G the gravitational constant. Differentiating both equations for secular changes, conserving angular momentum, and combining gives

$$d\varepsilon/dt = [m/(M+m)] \, (n/\omega_z) \, (MR^2/C) \, (a/R)^2 \sin \varepsilon \, (1-e^2)^{1/2} \cos i \, [da/dt / 2a - e \, de/dt / (1-e^2) - \tan i \, di/dt]$$

$$d(C\omega_z)/dt / C\omega_z = -\cot \varepsilon \, d\varepsilon/dt \tag{27}$$

where n is the lunar mean motion and R the Earth's radius, Evaluating with the tidal changes in the lunar orbit (dominated by da/dt) gives $d\varepsilon/dt = 19.6 \times 1@$ "/cy and

$d(C\omega_z)/dt \, / \, C\omega_z = -2.20 \times 10^{-8}$ /cy. The latter causes 110×10^{-6} "/cy^2 in (he precession. Both C-A and the deviation of C from the mean moment depend on the square of the Earth's rotation rate. Thus the value of $d\omega_z/dt \, / \, \omega_z = -2, 19 \times 10^{-8}$ /cy, 0.44% less than $d(C\omega_z)/dt \, / \, C\omega_z$. The change in C-A causes a precession change of -220×10^{-6} "/cy^2. This tidal despinning of the Earth by (he Moon causes changes in both lunar- and solar-induced precession. The solar tides also act to despin the Earth. The solar torque is much less well known than the lunar. It is a common approximation to assume that the ratio of solar to lunar torques is proportional to the square of the ratio of tide heights (0,46), though there is some evidence for a smaller ratio (Brosche & Wunsch 1990). Here the factor 1.21 is used to amend the lunar calculations for the solar contribution: the tidal obliquity rate is 24×10^{-4} "/cy and the tidal precession change is $(-102 - 1.21 \times 110) \times 10^{-6}$ "/cy^2 = -235×10^{-6} "/cy^2. A related, but not identical, calculation of the obliquity rate by Kaula (1964), when adjusted for recent secular acceleration measurements and the solar contribution, gives an obliquity rate of 17×10^{-d} "/cy.

There are a host of nontidal processes which change the spin rate of the Earth by exchanging angular momentum between the liquid core, solid mantle plus crust, oceans, and atmosphere, but these leave $C\omega_z$ unaffected. However, some of these processes do affect the precession through changes in C–A. The Earth's gravitational harmonic J_2 is proportional to C–A and exhibits a small secular decrease which has been detected from the analyses of ranges to the Lageos and Starlette satellites (Yoder et al. 1983, Rubincam 1984, Cheng et al. 1989, Watkins & Eanes 1993, Nerem et al. 1993). Consequently the precession rate should also exhibit a decrease. The J_2 rates from these studies lie in the range of $(-2,5$ to $-3.6) \times 10^{-9}$ /cy; they induce a sizable precession rate change in the range of $(-11.6$ to $-16.8) \times 1\&3$ "/cy^2. This is about 0.7% of the -2 "/cy^2 classical acceleration induced by ecliptic motion (next section) and two orders-of-magnitude larger than tidally induced accelerations. Though seeming to vary on thousand year time scales, the nontidal acceleration of the Earth's spin (Stephenson & Morrison 1984, 1985) appears to be in accord with the reported J_2 rates. While the J_2 rate is clearly visible in satellite data from the last one and one-half decades, there appear to be rate irregularities and questions about the separation of 18.6-yr tidal signatures which limit knowledge of the long-time average (Watkins & Eanes 1993). Watkins and Eanes found that the J_2 variations were fit better with 18.6-yr terms than with a linear J_2 change. There seem to be two choices: the reported J_2 rate is not linear, perhaps due to 18.6-yr tides which deviate from the expected, and the explanation for the nontidal acceleration of spin must be abandoned or Watkins and Eanes' results indicate decade-scale signatures in addition to the linear change with the better 18.6-yr fits explained as due to the data span being shorter than 18.6 yr plus the advantage of an extra parameter in the fit. For Table 4 the J_2 rate has been accepted as valid; adopting a value of -3×10^{-9} /cy yields -0.014 "/cy^2 in precession. This choice will give a precession acceleration valid since 1976, but the future extrapolation is uncertain. The J_2 rate uncertainty is the largest recognized uncertainty in the acceleration of precession. The precession is only sensitive to long-time changes in J_2 and the appropriate contribution to the precession will depend upon further monitoring of J_2 changes with artificial satellites.

8. Polynomials

The IAU precession paper (Lieske et al. 1977) gives polynomials in time for orienting the Earth based on the IAU-adopted general precession rate, obliquity, and masses. Two equivalent matrix formulations have also been published (Lieske 1979, Fabris 1980).

improved ecliptic motion led to updates by Bretagnon & Chapront (1981), Laskar (1986), and Simon *et al. (1993).* The latter paper includes improved precession rate, obliquity, and masses as well. In this paper theoretical contributions to precession and obliquity rates and higher derivatives have been identified and computed, In this section the theoretical improvements plus updated values for precession rate, obliquity, masses, and ecliptic motion will be used to generate revised polynomial expressions.

In this section the notation of the lAU paper is used except that the (iIda has been dropped, The subscript A denotes the accumulated quantity. Thus Ψ_A and P_A are the accumulated luni-solar and general precession, respectively. The expressions will be derived for a single time argument for use with the J2000 epoch. The fixed ecliptic and equator planes of J2000 and the moving ecliptic and equator of date constitute the basic geometry. See the I AU paper and Fig. 1 for the definition of the variables,

The basic differential equations were given in the lAU paper but they require additional terms due to the obliquity rate contributions with respect to inertial space, The obliquity and precession rates and derivatives of Table 4 use a coordinate frame which is moving with the equinox. The total contribution of the obliquity rate (with respect to space, not the moving ecliptic) from the table is denoted R_ε and the total contribution to the precession rate multiplied by sin ε_A is denoted R_ψ These two components of the rotation vector are in the plane of the moving equator, Two of the differential equations are just the projections of these two rates through the angle χ_A ("planetary precession") between the moving equinox and the intersection of the fixed ecliptic and moving equator.

$$d\omega_A/dt = \cos \chi_A\ R_\varepsilon + \sin \chi_A\ R_\psi$$

$$\text{Sin } \omega_A\, d\psi_A/dt = COS\ \chi_A\, R_\psi\ \sin \chi_A\, R_\varepsilon \tag{28}$$

The differential equation for the obliquity rate with respect to the moving ecliptic involves both the motion of the ecliptic and R_ε

$$d\varepsilon_A/dt = \cos p_A\, dQ'/dt - \sin p_A\, dP'/dt + (1-\cos \pi_A) \cos(\Pi_A+p_A)\, d\pi_A/dt + R_\& \tag{29}$$

where $P' = \sin \pi_A \sin \Pi_A$ and $Q' = \sin \pi_A \cos \Pi_A$ describe the ecliptic pole with Π_A and π_A being the node and inclination, respectively, of the moving ecliptic on the fixed ecliptic.

In addition to P' and Q', which are input functions for the ecliptic motion, the right-hand sides of the differential equations are functions of χ_A and P_A. Two geometrical equations are needed to link these latter two variables with the others.

$$\sin \chi_A = \sin \pi_A \sin(\Pi_A+p_A) / \sin \omega_A$$

$$\cos(\Pi_A+p_A) = \cos \chi_A\ \cos(\Pi_A+\psi_A) + \sin \chi_A\ \sin(\Pi_A+\psi_A) \cos \omega_A \tag{30}$$

The three differential equations and the two geometrical equations must be solved simultaneously for $\omega_A, \psi_A, \varepsilon_A, \chi_A$, and p_A.

The simultaneous solution was performed with a numerical technique. The variables are represented with polynomials of time. The five equations are evaluated at equal time intervals, the polynomial coefficients are fit, and the procedure is iterated to convergence.

Input quantities are the J2000 obliquity (ε_0) and general precession rate plus the polynomials for P' and Q'. The constant PO in Table 4 is determined from the initial obliquity and precession rate and (he other precession rates in the table. At J2000 the. rates of general and luni-solar precession are linked through the rate of the planetary precession χ_A projected on the ecliptic plane. The iterative solution for the polynomials was done in extended precision on a microcomputer.

parameters such as the commonly used ζ_A, θ_A, and z_A are derived geometrically from the above set of variables. Numerical polynomial fits were also used. Two points are noted for generating the polynomials for ζ_A and z_A. The polynomial for ω_A is carried to one higher degree than those for ζ_A and z_A. It is necessary to include a constant in ζ_A and z_A when the zero coefficient oft in ω_A is finite. The constant is $d\omega_A/dt / \sin \varepsilon_0 \, d\psi_A/dt$ evaluated at J2000 and has opposite signs for ζ_A and z_A.

The program was able to successfully reproduce the polynomials in the IAU paper with a deviation of no more than 1 in the last digit except for Π_A. As also experienced by Fabri (1980), in the IAU paper the number of digits in Π_A exceeds those in P' and Q' and apparently additional digits in P' and Q' were used there.

Greenwich Mean Sidereal Time (GMST) is referenced to a moving equinox. Consequently, the IAU-adopted polynomial expression for GMST given by Aoki *et al.* (1982) is specific to the lAU precession theory, As pointed out by Williams & Melbourne (1982) and Zhu & Mueller (1983), changing the precession expressions without changing the GMST expression will result in a change in UT1 determination. Consequently, an additional equation has been evaluated numerically, The fundamental parameter is the rotation rate of a rigid Earth with respect to inertial space about it's symmetry axis

$$d(GMST + \chi_A) / dt - COS \; \omega_A \, d\psi_A/dt \tag{31}$$

with due consideration for the units. The nonlinear parts of the GMST expression at zero hour UT, GMST = GMSTO + UT1, come from

$$\int \cos \omega_A \; d\psi_A/dt \; dt - \chi_A \tag{32}$$

, dividing arcseconds by 15 to convert to seconds. The coefficients of the constant and linear terms were set by imposing the condition that at J2000 there would not be a discontinuity of UT1 (the constant coefficient matches Aoki *et al.*), its rate, or the rotation rate of the Earth in space. In Aoki et *al.* the constant and linear coefficients were picked for continuity of UT 1 determined from optical astrometric measurements of catalog stars rather than continuity with respect to an inertial frame. Inertially referenced techniques now dominate the determination of UT1 so there is no counterpart to the catalog equinox drift, It is conventional to derive the small nonlinear terms of GMSTO using a linear time scale for the independent time, but to evaluate the entire GMSTO expression using a UT1 time scale.

Since the IAU theory for precession appeared, there have been improvements in the computation of the motion of the ecliptic due to theoretical advances and improved planetary masses, better measured values for precession rate and obliquity, and the addition of the theoretical adjustments of this paper. To illustrate the resulting changes, revised expressions are presented here, The values here match those used to generate Table 3 (section 5). The ecliptic motion is taken from Simon *et al.* (1993) including planetary mass corrections, The theoretical adjustments of Table 4 have been used. The

resulting expressions are given in Table 5. The units are arcseconds and Julian centuries measured from J2000 [t = (JD - 2451545.0) / 36525], except for GMST which uses seconds and centuries of UT1 measured from JD 2451 S45.0 UT1 = 12 hr. UT] on January 1, 2000. While these expressions can serve those who need the highest accuracy now, it should be anticipated that there will be future improvements: some theoretical, certainly in the measurement{ of the precession constant and obliquity, and hopefully in the predictive knowledge of the J2 rate.

The polynomial expressions in Table 5 can be used for times extending out to a few millennia, but for longer times they are just representations of an average precession rate and obliquity plus long-periodic, or at least quasi-periodic, terms with periods exceeding 10,000 yr (Berger 1976, Laskar *et al*. 1993). For millions of years the small tidal acceleration is inexorable and modifies the precession and obliquity behavior for ancient times (Berger *et al*. 1992). Nontidal J2 change must be more transient since most of the Earth's oblateness is controlled by its spin; large deviations of J2 from equilibrium would cause large stresses.

9. Rotations

Considered in this section are the rotations suitable for the various sets of precession parameters in Table 5. When combining the rotations for precession and nutations, the number of rotations can be minimized. Finally, an expedient procedure is given which is suitable for introducing the most important (linear) corrections to precession and obliquity without undertaking the more extensive and complete modifications,

Consider the rotations which can be used to orient the Earth for precession and nutation. The standard procedure is to precess from the mean equator and equinox of .12.000 to the mean equator and equinox of date using the angles ζ_A, θ_A, and z_A and then to nutate to the true equator and equinox of date by rotating into the mean ecliptic of date, applying nutation in longitude to reach the true equinox of date, and then to rotate to the true equatorial plane including nutation in obliquity. The sequence of six rotations (R_i is the rotation matrix around axis i) is

$$R1(-\varepsilon_A - \Delta\varepsilon)\, R3(-\Delta\psi)\, R1(\varepsilon_A)\, R3(-90°-z_A)\, R1(\theta_A)\, R3(90°-\zeta_A) \qquad (33)$$

An alternative is to precess by moving along the fixed ecliptic to the intersection with the mean equator of date, then rotate along that equator to the mean equinox of date, and then to nutate as before. The seven rotations are

$$R1(-\varepsilon_A - \Delta\varepsilon)\, R3(-\Delta\psi)\, R1(\varepsilon_A)\, R3(\chi_A)\, R1(-\omega_A)\, R3(-\psi_A)\, R1(\varepsilon_0) \qquad (34)$$

A second alternative is to precess by moving along the fixed ecliptic to the intersection with the mean ecliptic of date, rotate back along the mean ecliptic, and then nutate.

$$R1(-\varepsilon_A - \Delta\varepsilon)\, R3(-\Pi_A - p_A - \Delta\psi)\, R1(\pi_A)\, R3(\Pi_A)\, R1(\varepsilon_0) \qquad (35)$$

The third procedure requires only five rotations. If one is to nutate as well as precess, it is possible to bypass the mean equator of date and combine the precession and nutation along the moving ecliptic. Five rotations is not the minimum for combining precession and nutations. With four rotations one can move along the fixed equator to the intersection with the moving equator (angle ξ_A), rotate into the moving equator (ε'_A),

combine the rotation along the ecliptic of date (η_A to the mean equinox of date) and nutation in longitude, and rotate to the true equator

$$R_1(-\epsilon_A-\Delta\theta)\,R_3(-\eta_A-\Delta\psi)\,R_1(\epsilon'_A)\,R_3(\xi_A)$$

The angles ξ_A, ϵ'_A, and η_A have not been given with conventional precession expressions in the past. They are illustrated in Fig. 1 and the expressions are given in Table S.

Note that for the changes in precession and obliquity polynomials not due to ecliptic motion, only the last two rotations in the last two cases would be changed; for the first two cases the precession and obliquity changes are distributed over multiple rotations, The last two cases make it clear that a change to the precession rate and the -0.0244 "/cy correction to the obliquity rate, and even the larger number of theoretical contributions in Table 4, could be added into the corresponding nutation parameters as an alternative way to introduce them (a similar conclusion was reached by Folkner *et al.* 1993 and VLBI fits to observations have often included linear terms in their "nutation" corrections). This is an expedient, short-term solution for those who do not wish to reprogram the precession and GMST expressions. For this expedient approach the equation of equinoxes associated with $\Delta\psi$ will automatically satisfy the concerns of Williams & Melbourne (1982) and Zhu & Mueller (1983) about precession modifications changing UT], so that GMST does not require revision. This expedient procedure does not work for geometrical revisions to ecliptic motion (purely P', Q', Π_A, and π_A but also parts of other parameters) since they appear in multiple rotations and tend to cancel, but it could be applied to the dynamical consequences of those revisions. However, it is observed that if an expedient procedure becomes too complicated then it is not expedient, It is best suited to easily inserting the linear corrections to precession and obliquity.

10. Nutation Corrections, Scaling, and Comparisons

The 1980 IAU nutation series (Seidelmann 1982) was a combination of the rigid-body series of Kinoshita (1977) and the elastic and structured-Earth corrections due to Wahr (1979, 198 1). Since the 1980 IAU nutation working group paper there have been two revisions of rigid-body nutations (Zhu & Groten 1989, Kinoshita & Souchay 1990). The nutation theories allowing for the Earth's elasticity and core are based on the rigid-body theories and it is well to compare and understand those rigid-body theories .

Zhu and Groten utilized the earlier work of Kinoshita (1977) extending the series to smaller terms, and adding both second-order terms and corrections for the Earth's J3. It has served as the basis for non-rigid body treatments by Zhu *et al. (1990),* and the several ZMOA series of Herring (1990), Mathews *et al. (1991),* and Herring et *al. (1991).* Kinoshita and Souchay also extended the series to smaller terms, and added second-order terms, J3 effects, and planetary terms involving planetary arguments. In addition they added small solar terms due to the offset of the Earth from the center of mass of the Earth-Moon system and revised the expression for the (C–A)/C scaling of the nutation series from the precession constant. In Kinoshita (1977) and Kinoshita and Souchay the J2 tilt effects on the scaling and 18.6-yr terms are present. The in-phase 18.6-yr nutation coefficients and the -0.0056 "/cy precession due to the planetary tilt effect is present in the former, but not the latter work, Kinoshita and Souchay also add -0.014 "/cy second-order contributions to the precession and make small revisions to the first-order contributions which are not present in the earlier works. Thus there are small differences in the scaling of Kinoshita and Souchay, Zhu and Groten, and this paper which are addressed below. Only Kinoshita and Souchay have the small center-of-mass offset

corrections, three of which have amplitudes of 0.02 mas in longitude. Comparison of the J3 contributions in the two papers shows poor agreement; the 3231 day obliquity coefficients have different signs and differ by 0.12 mas. In addition, the 6164 day coefficients disagree by a factor of two. See Souchay (1993) for further comparisons.

The coefficients of Kinoshita and Souchay's rigid-body nutation theory would have to be multiplied by 0,99993782 (section 5) to match the precession rate and other changes of this paper. This would cause the 18.6-yr $\Delta\psi$ coefficient to increase by 1,075 mas and the 18.6-yr $\Delta\varepsilon$ coefficient to change by -0.574 mas. These corrections are in addition to those of Table 1 and taken together the in-phase corrections to the 18.6-yr coefficients are 1.045 mas in $\Delta\psi$ (giving -17.28076") and --(1.57 1 mas in $\Delta\varepsilon$ (9.22800").

To match the constants of this paper the coefficients of Zhu and Groten's rigid-body nutation series need to be multiplied by 0,9999308 for the lunar terms and 0.9999297 for the solar terms. For the in-phase 18.6 yr coefficients this gives –17.28075" in $\Delta\psi$ and 9.22792" in $\Delta\varepsilon$. Thus after correction to a common $(C–A)/C$ and compensation for the planetary tilt effect, the 18.6-yr terms of Zhu and Groten and Kinoshita and Souchay differ by only 0.01 mas in longitude and 0.08 mas in obliquity.

To all nutation series since Woolard (1953) the out-of-phase planetary tilt contributions of Table 1 need to be added. To all nutations series since Woolard, a -0,15 rnas annual term from the yearly variation of the geodesic precession needs to be added to the nutation in longitude (Voinov 1988, Gill *et al.* 1989, Fukushima 1991). For highest accuracy, nutation terms with planetary arguments, such as those of Kinoshita and Souchay, should also be included, While it causes minor changes in the resulting nutation series evaluation, the arguments of the 1980 IAU series and other series can be improved upon by using the values of Simon *et al.* (1993). The annual argument (ℓ') differs by 5" at J2000, but the values of ℓ' and L' depend very much upon which long-period terms are being carried when these arguments are fit to the time-varying heliocentric orbit. It should be compatible with the formulation used to generate the nutations with planetary arguments.

11. Summary

Improvements in the accuracy of the observed motion of the Earth's equator plane and the wish to use these observations to infer the Earth's properties make improvements in the theories of precession, obliquity rate, and nutations desirable. The rate terms computed in this paper come from lunar orbit perturbations due to the planets and the Earth's J2 plus direct planetary torques on the Earth.

The corrections to the obliquity rate are due to direct planetary torques on the Earth (see section 4 and Table 2), torques due to planetary perturbations on the lunar orbit (section 3), and tidal influences (section 7, Table 4), Together these corrections are -0.244 milliarcseconds/yr (mas/yr). This correction is a motion in space; the conventional -0.468 "/yr obliquity rate is due solely to ecliptic motion, not to changes in the Earth's orientation, The IAU-adopted theory of precession and obliquity changes requires correction for this contribution to the obliquity rate, The largest contribution to the obliquity rate in space was earlier computed by Woolard (1953), but its reality was questioned by Kinoshita (1977). Section 6 discusses the reason for this discordant interpretation and concludes that the rate is real. The obliquity motion in space should be observable by the very long baseline interferometry and lunar laser ranging techniques.

In addition to the obliquity rate amendments, there are small contributions to (he precession rate due to direct planetary torques and lunar orbit effects (sections 3 and 4, Table 2). The sum of the various contributions to obliquity and precession rates is given in Table 3 (section 5). Based on recent measurements a general precession rate of 50,2877 "/yr at J2000 was adopted. For a rigid Earth this corresponds to the moment-of-inertia combination $(C-A)/C = 0.0032737634$. Combined with a satellite-determined J2 this gives a normalized polar moment of inertia $C/MR^2 = 0.3307007$ and a normalized mean moment $I/MR^2 = 0.3299789$ with R the equatorial radius.

The contributions to obliquity and precession rates are not constant with time and the higher derivatives from these and other sources are computed in section 7, Table 4 summarizes both linear and nonlinear (in time) contributions. The theory for orienting the Earth (precession, obliquity changes, and Greenwich Mean Sidereal Time) is considered in section 8 and revised polynomial expressions are presented (Table 5). In addition to the theoretical corrections of this paper, these expressions use improved values of the obliquity, precession rate, masses, and ecliptic motion.

Matrix rotations which combine precession and nutation are considered in section 9. The conventional rotation scheme is not optimized for the number of rotations, A sequence of four rotations is given which incorporates both precession and nutation.

The torques, due to lunar orbit perturbations from the planets, also give rise to nutation contributions (section 3 and Table 1), The largest contributions are to the 18.6-yr nutations: $-0.030 \sin \Omega + 0.137 \cos \Omega$ to $\Delta\psi$ (in mas) and $-0.028 \sin \Omega + 0.003 \cos \Omega$ to $\Delta\varepsilon$. The small out-of-phase corrections arise because, other than the Earth, the planetary nodes on the ecliptic are not aligned with the dynamical equinox. Out-of-phase nutations are conventionally considered to only arise from energy dissipation in the Earth and oceans, but these are exceptions.

The torques which cause precession and nutation depend on $(C-A)/C$ so that an accurate determination of the precession rate sets the scale of the nutations. This scaling of the two most recent rigid-body theories is discussed in section 10. Also discussed are the additions appropriate to each of these nutation theories.

Since the IAU expressions for precession and nutations were adopted, both theoretical improvements and refined measurements have become available. The theoretical contributions of this paper may be added to revised computations of ecliptic motion, rigid-body nutations, dissipative effects in the Earth's interior and oceans, and relativistic effects, Improved measurements of the precession constant and individual nutation terms are available. The latter have permitted refined computations of the non-rigid-body contributions to nutations. Understanding of the orientation of the Earth's equator and the fundamental influences on the orientation is advancing.

REFERENCES

Aoki, S., Guinot, B., Kaplan, G. H., Kinoshita, H., McCarthy, D. D., & Seidelmann, P. K. 1982, Astron. Astrophys. 105, 359

Berger, A. 1976, Astron. Astrophys. 51, 127

Berger, A., Loutre, M. F., & Laskar, J. 1992, Science 255,560

Bretagnon, P. 1982, Astron. Astrophys. 114,278

Bretagnon, P., & Chapront, J. 1981, Astron. Astrophys. 103, 103

Brosche, P., & Wunsch, J. 1990, in *Earth's Rotation from Eons to Days,* edited by P. Brosche and J. Sundermann, Springer-Verlag, Berlin, 141

Chapront-Touze, M., & Chapront, J. 1983, Astron. Astrophys. 124,50

Chapront-Touze, M., & Chapront, J. 1988, Astron. Astrophys. 190,342

Chariot, P., Severs, O. J., Williams, J. G., & Newhall, X X 1991, in *Proceedings of the 127th Colloquium of the International Astronomical Union, Reference* Systems, edited by J. A. Hughes, C. A. Smith, and G. H. Kaplan (U. S. Naval Ohs., Washington), p. 228

Cheng, M., Eanes, R., Shum, C., Schutz, B., & Tapley, B. 1989, Geophys. Res. Lett. 16, 393

Dehant, V. 1988, in *The Earth's Rotation and Reference Frames for Geodesy and Geodynamics,* edited by A. K. Babcock and G. A. Wilkins, Kleuwer, Dordrecht, 323

Dehant, V. 1990, Geophys. J. Royal Astron. Sot. 100,477

Fabri, E. 1980, Astron. Astrophys. 82, 123

Folkner, W. M., Chariot, P., Finger, M. H., Newhall, X X, Williams, J. G., Severs, O. J., & Standish, E. M. 1993, submited to Astron. Astrophys.

Fukushima, T. 1991, Astron. Astrophys. 244, L11

Gill, E., Soffel, M., Ruder, H., & Schneider, M. 1990, in *Earth's Rotation from Eons to Days,* edited by P. Brosche and J. Sundermann, Springer-Verlag, Berlin, 13

Herring, T. A. 1991, in *Proceedings of the 127th colloquium of the International Astronomical Union, Reference Systems,* edited by J. A, Hughes, C. A. Smith, and G. H. Kaplan (U. S. Naval Ohs., Washington), p. 157

Herring, T. A., Buffet, B. A., Mathews, P. M., & Shapiro, 1.,1.1991, J. Geophys. Res. 96, 8259

Kaula, W. M. 1964, Reviews of Geophys. 2,661

Kinoshita, H. 1975, *Smithsonian Astrophysical observatory Special Report, No. 364*

Kinoshita, H. 1977, Celes. Mech. 15,277

Kinoshita, H., & Souchay, J. 1990, Celes. Mech. and Dyn. Astron. 48, 187

Laskar, J. 1986, Astron. Astrophys. 157, 59

Laskar, J., Joutel, F., & Boudin, F. 1993, Astron. Astrophys. 270, 522

Lieske, J. H. 1979, Astron. Astrophys. 73,282

Lieske, J. H., Lederle, T., Fricke, W., & Morando, B, 1977, Astron. Astrophys. 58, 1

Marsh, J. G., Lerch, F. J., Putney, B. H., Felsentreger, T. L., Sanchez, B. V., Klosko, S. M., Patel, G, B., Robbins, J. W., Williamson, R.G., Engelis, T. L., Eddy, W. F., Chandler, N. L., Chinn, D. S., Kapoor, S., Rachlin, K. E., Braatz, L. E., & Pavlis, E. C. 1990, J. Geophys. Res. 95,22043

Mathews, P. M., Buffett, B. A., Herring, T. A., & Shapiro, 1.1.1991, J. Geophys. Res. 96, 8243

McCarthy, D. D., & Luzum, B. J. 1991, Astron. J., 102, 1889

Miyamoto, M., & Soma, M. 1993, Astron. J. 105,691

Nerem, R. S., Chao, B. F., Au, A. Y., Chan, J, C., Klosko, S. M., Pavlis, N. K., & Williamson, R. G. 1993, Geophys. Res. Lett. 20,595

Newhall, X X, Standish, E. M., & Williams, J. G. 1993, DE 242 ephemeris of the planets and Moon

Rubincam, D. P. 1984, J. Geophys. Res. 89, 1077

Seidelmann, P. K. 1982, Celestial Mech. 27,79

Simon, J. L., Bretagnon, P., Chapront, J., Chapront-Touze, M., Francou, G., & Laskar, J. 1993, Astron. Astrophys., in press

Souchay, J. 1993, Astron. Astrophys. 276,266

Souchay, J., & Kinoshita, H. 1991, Celes. Mech. and Dyn. Astron. 52,45

Standish, E. M. 1982, Astron. Astrophys. 114,297

Stephenson, F. R., & Morrison, 1.. V. 1984, Phil. Trans. Royal Soc. Lend. A313, 47

Stephenson, F. R., & Morrison, L. V. 1985, Geophys. Surveys 7,201

Steppe, J. A., Oliveau, S. H., & Severs, O. J. 1993, IERS Annual Report, in press

Voinov, A, V. 1988 Celes. Mech. 42,293

Wahr, J, M. 1979, Ph. D. thesis, Univ. of Colorado, Boulder, Colorado

Wahr, J. M. 1981, Geophys. J. Royal Astron. Sot. 64,705

Wahr, J. M., & Bergen, Z. 1986, Geophys. J. Royal Astron. Sot. 87,633

Wahr, J. M., & Sasao, T. 1981, Geophys. J. Royal Astron. Sot, 64, 747

Watkins, M., & Eanes, R. 1993, in *Advances in Space Research, vol.* 13, *No.* 11, 251

Williams, J. G., & Melbourne, W. G. 1982, in *Proceedings, High-Precision Earth Rotation and Earth-Moon Dynamics: Lunar Distances and Related Observations,* edited by O. Calame, D. Reidel, Hingham, Mass., 293

Williams, J. G., Newhall, X X, & Dickey, J. O. 1991, Astron. Astrophys. 241, L9

Williams, J. G., Newhall, X X, & "Dickey, J. 0.1993, in *Contributions of Space Geodesy to Geodynamics: Earth Dynamics, Geophysical Monograph of the American Geophysical Union,* Vol. 24, edited by D. E. Smith and D. L. Turcotte, Washington, D. C., in press

Woolard, E. W. 1953, in *Astronomical Papers for the American Ephemeris and Nautical Almanac* XV, Pt. 1, (U. S. Govenrment Printing Office, Washington)

Yoder, C. F., Williams, J. G., Dickey, J. O., Schutz, B. E., Eanes, R. J., & Tapley, B. D. 1983, Nature 303,757

Zhu, S. Y., & Groten, E. 1989, Astron. J. 98, 1104

Zhu, S. Y., Groten, E., & Reigber, C. 1990, Astron. J. 99, 1024

Zhu, S. Y., & Mueller, I. I. 1983, Bull. Geod. 57,29

FIGURE CAPTION

FIG. 1. Relation between the fixed equator (mean equator) and fixed ecliptic of J2000 and the moving (mean of date) equator and ecliptic. The arc from the moving equinox to the node of the moving ecliptc on the fixed ecliptic is $AA = \Pi_A + p_A = \phi_A + \eta_A$.

TABLE 1. Nutation terms due to J_2 and planetary tilt effects
of lunar orbit. Lunar mean longitude is $L = \Omega + F$.

argument	$\Delta\psi$		$\Delta\varepsilon$	
	sin	cos	sin	cos
	mas	mas	mas	mas
J_2 Tilt				
Ω	-1,4782	0,0000'	0,0000	0.1557
$2*\Omega$	0,0049	0.0000	0,0000	-0,0026
$2*L$	0,0151	0.0000	0,0000	-0,0081
Planetary				
Ω	-0,0301	0.1366	-0,0277	0.0029
$2*\Omega$	-0,0005	0.0060	0,0032	0,0003
$2*L$	0,0003	-0,0028	-0,0015	-0,0002

TABLE 2. Precession and obliquity rates from
direct planetary torques on the Earth's bulge,

Planet	ψ rate mas/yr	ε rate mas/yr
Mercury	0,003651	-0,000090
Venus	0. 18?273	-0,017372
Mars	0,005393	0,000255
Jupiter	0,116665	0,002782
Saturn	0,005177	0,000217
Uranus	0,000100	0,000001
Neptune	0,000029	0,000001
Total	0,318287	-0,014207

TABLE 3. Contributions to precession and obliquity rates,
(C-A)/C = 0.0032737634 and obliquity 23°26'21,409" at J2000.

Contribution	prec. rate arcsec/yr	obliq. rate arcsec/yr
Sun first order	15, 9488?0	
Moon first order	34, 457698	
Second order	-0, 000468	
J_4	0, 000026	
Tilt effects	-0, 002686	-0, 000254
Direct planetary	0, 000318	-0, 000014
Tidal		0, 000024
Geodesic precession	-0, 019194	
Total space motion	50, 384565	-0, 000244
Ecliptic motion	-0, 096865	-0, 468096
General motion	50, 287700	-0, 468340

TABLE 4. Time and obliquity dependence of precession and obliquity rates ("/century) which are needed to calculate the evolution of precession and obliquity with time.

Source	Rate in "/century	ε Dependence
	Precession	
Luni-solar, direct planetary torque	$P_0 \cos \varepsilon_0 - 0.003395\ t - 6 \times 10^{-6}\ t^2$	$\cos \varepsilon$
Geodesic precession	$-1.919362 + 2.7 \times 10^{-6}\ t$	1
Second order (M_3)	-0.03310	$6 \cos^2 \varepsilon - 1$
Second order	-0.01368	$3 \cos^2 \varepsilon - 1$
J_4 precession	+0.00260	$\cos \varepsilon\ (4-7 \sin^2 \varepsilon)$
J_2 tilt	-0.2630	$\cos 2\varepsilon / \sin \varepsilon$
Planetary tilt and direct torque	-0.00643+0.001074 t	$\cos 2\varepsilon / \sin \varepsilon$
Tides on lunar orbit	-0.000102 t	$\cos^2 \varepsilon$
Tides on spin and moments	-0.000133 t	$\cos^3 \varepsilon$
J_2 rate	-0.0140 t	Cos &
	Obliquity	
Planetary tilt and direct torque	$-0.0268-0.000044\ t + 3 \times 10^{-6}\ t^2$	$\cos \varepsilon$
Tides	+0.0024	$\sin \varepsilon \cos \varepsilon$

TABLE 5. Polynomial expressions for orientation of the Earth's equator (arcsec). Time t in Julian centuries from J2000 (JD 245 1545.0). Greenwich mean sidereal time (seconds) at O hours UT1. Time in UT1 centuries from 12 hr. UT1, JD 2451545.

Angle	constant	t	t^2	t^3	t^4
P'	0. 000000	⁻4, 199610	0, 193971	-0, 000223	-0, 000001
Q'	0, 000000	--46, ?09560	0, 051043	0, 000522	-0, 000001
π_A	0, 000000	46, 997570	-0, 033506	-0, 000124	0(000000
Π_A	629543, 967373	-867, 919986	0. 153382	0, 000026	-0, 000004
p_A	0. 000000	5028 , 770000	1. 105407	0, 000076	-0, 000024
ψ_A	0.000000	5038, 456501	-1, 078977	-0, 0011' 11	0, 000133
ω_A	. 84381 , 409000	-0, 024400	0, 051268	-0, 007727	0, 000000
χ_A	0.000000	10, 557700	-2, 381366	-0, 001208	0, 000170
ε_A	84381,409000	-' 16, 833960	-0, 000174	0, 002000	-0, 000001
ζ_A	2, 511180	2306, 071060	0, 299027	0, 018017	-0, 000005
z_A	-2, 511180	2306, 065079	1, 092516	0, 018265	-0, 000029
θ_A	0.000000	2004, 182023	-0, 429466	-0, 041822	-0, 000007
ξ_A	0, 000000	10, 557700	0, 493164	-0, 000309	-0, 000003
ε'_A	84381, 409000	-46, 809560	0, 051142	0, 000531	0, 000000
η_A	0. 000000	5038.456501	1, 558353	-0, 000186	-0, 000027
GMST0	24110, 54841	8640184, ?928613	0, 0927695	-0, 0000003	-0, 0000020

FIG. 1. Relation between the fixed equator (mean equator) and fixed ecliptic of J2000 and the moving (mean of date) equator and ecliptic, The arc from the moving equinox to the node of the moving ecliptc on the fixed ecliptic is $AA = \Pi_A + p_A = \sigma_A + \eta_A$.